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In-plane perturbation of the tunnel-crack under shear loading II: Determination of the fundamental kernel

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Abstract

For any plane crack in an infinite isotropic elastic body subjected to some constant loading, Bueckner–Rice's weight function theory gives the variation of the stress intensity factors due to a small coplanar perturbation of the crack front. This variation involves the initial SIF, some geometry independent quantities and an integral extended over the front, the "fundamental kernel" of which is linked to the weight functions and thus depends on the geometry considered. The aim of this paper is to determine this fundamental kernel for the tunnel-crack. The component of this kernel linked to purely tensile loadings has been obtained by Leblond et al. [Int. J. Solids Struct. 33 (1996) 1995]; hence only shear loadings are considered here. The method consists in applying Bueckner–Rice's formula to some point-force loadings and special perturbations of the crack front which preserve the crack shape while modifying its size and orientation. This procedure yields integrodifferential equations on the components of the fundamental kernel. A Fourier transform in the direction of the crack front then yields ordinary differential equations, that are solved numerically prior to final Fourier inversion.

Keywords: Tunnel-crack; Shear loading; Mode 2; Mode 3; Crack front; In-plane perturbation; Fundamental kernel; Bueckner–Rice theory; Weight functions; Integrodifferential equations; Fourier transform

1. Introduction

Consider a plane crack embedded in an infinite isotropic elastic solid subjected to some arbitrary, constant loading. Eq. (1) of Part I gives the variation of the SIF resulting from any small, in-plane perturbation of the crack front. This equation notably involves an integral extended over the front. As will be detailed below, the "fundamental kernel" **Z** in this integral is linked to the weight functions of the crack, i.e. to the SIF induced on the crack front by unit point forces exerted on the crack lips, in the limit when the points of application of these forces get infinitely close to the crack front. Therefore, it depends upon the entire geometry of the crack. It can be deduced for instance from the works of Bueckner (1987) or Meade and Keer (1984) on weight functions for a half-plane crack, from those of Kassir and Sih (1975), Tada et al.

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(1973) and Stallybrass (1981) for a circular connection and from that of Bueckner (1987) for a penny-shaped crack. This has been done by Rice (1985) and Gao and Rice (1986) for a half-plane crack, by Gao and Rice (1987a) for a circular connection and by Gao and Rice (1987b) and Gao (1988) for a penny-shaped crack. The weight functions and the fundamental kernel are also known for an interface half-plane crack (Lazarus and Leblond, 1998). For the tunnel-crack, the component Z_{11} linked to *tensile* loadings has been derived by Leblond et al. (1996). The aim of Part II of this work is to derive the other components Z_{mn}, $m, n = 2, 3$ linked to *shear* loadings. [1]

The method used is based on both works of Leblond et al. (1996) and Lazarus and Leblond (1998). It is of "special" rather than "general" nature in the terminology employed by Bueckner (1987). This means that it avoids the calculation of the entire solution of the elasticity problems implied, but concentrates instead on the sole feature of interest, namely the distribution of the SIF along the crack front. Considering the complexity of the equations obtained even with such a "reduction", one may reasonably conjecture that any "general" method of solution would be intractable.

The principle of our method is to apply Eq. (1) of Part I to some special loadings and perturbations of the front. The loadings considered consist of point forces applied close to the crack front, so that the SIF prior to the perturbation are just components of the fundamental kernel **Z**. The perturbations envisaged consist of a small translation of the rear part of the front and a small rotation of both parts about an axis normal to the crack plane. Since the shape of the crack is preserved in these transformations, the new SIF are still connected to **Z**. One thus obtains integrodifferential equations on the components of **Z** which can be solved through Fourier transform along the direction of the crack front.

The paper is organized as follows. Necessary elements from Part I are recalled in Section 2 for completeness. The integrodifferential equations on the components of **Z** are presented in Section 3. The Fourier transform of these equations in the direction of the crack front yields second order ordinary differential equations, duly completed by suitable "initial" conditions. This complete system of equations is presented in Section 4, and the necessary subsequent Fourier inversion is sketched in Section 5. As could be forecast in view of the complexity of the problem, this system does not have any simple analytical solution and must be solved numerically. The numerical procedure and results are presented in the final Section 6.

2. Preliminaries

2.1. Elements of Part I

For ease of reference, indispensable elements of Part I are briefly recalled here.

For the tunnel-crack of width $2a$, the fundamental kernel **Z**, which depends a priori on three parameters, a, z, z', can be expressed in terms of two operators **f** and **g** depending only on one parameter through the following relations:

$$\mathbf{Z}(a; z^+, z'^+) = \mathbf{Z}(a; z^-, z'^-) \equiv \frac{\mathbf{f}((z'-z)/a)}{(z'-z)^2} \tag{1}$$

$$\mathbf{Z}(a; z^+, z'^-) = \mathbf{Z}(a; z^-, z'^+) \equiv \frac{\mathbf{g}((z'-z)/a)}{a^2} \tag{2}$$

Moreover, the following relations hold:

[1] As will be seen, all other remaining components are zero.

$$f_{12} = f_{21} = f_{13} = f_{31} = g_{12} = g_{21} = g_{13} = g_{31} \equiv 0 \tag{3}$$

$$f_{32} = -(1-\nu)f_{23}, \quad g_{32} = -(1-\nu)g_{23} \tag{4}$$

Hence, the determination of the fundamental kernel is reduced to that of eight scalar functions of one variable: f_{11}, f_{22}, f_{33}, f_{23}, g_{11}, g_{22}, g_{33}, g_{23}.

The functions f_{11} and g_{11} pertaining to mode 1 loadings have been calculated by Leblond et al. (1996). The aim of this work is to determine the six other functions, pertaining to shear mode $2+3$ loadings. Among them, f_{22}, f_{33}, g_{22}, g_{33} are even and f_{23}, g_{23} odd functions.

Our starting point is Eq. (1) of Part I, which takes the form (10) and (11) for δK_2 and δK_3 in the case of a tunnel-crack. In a compact form more suitable here, these equations read:

$$\delta K_m(z^+) = [\delta K_m(z^+)]_{\delta a(z'^{\pm}) \equiv \delta a(z^+)} + N_{mn} K_n(z^+) \frac{\mathrm{d}\delta a}{\mathrm{d}z}(z^+) + \mathrm{PV} \int_{-\infty}^{+\infty} f_{mn}\left(\frac{z'-z}{a}\right) K_n(z'^+) \frac{\delta a(z'^+) - \delta a(z^+)}{(z'-z)^2} \mathrm{d}z' + \int_{-\infty}^{+\infty} g_{mn}\left(\frac{z'-z}{a}\right) K_n(z'^-) \frac{\delta a(z'^-) - \delta a(z^+)}{a^2} \mathrm{d}z', \quad m,n = 2,3 \tag{5}$$

where Einstein's implicit convention is used for the index n and

$$N_{23} = -\frac{2}{2-\nu}, \quad N_{32} = \frac{2(1-\nu)}{2-\nu} \tag{6}$$

ν denoting Poisson's ratio and other components of $\mathbf{N}$ being zero. The values of the $\delta K_m(z^-)$ for a point $M^-(z^-)$ belonging to the line $(x=-a)$ are given by the same expression with the obvious substitutions $z^+ \to z^-$, $z'^{\pm} \to z'^{\mp}$.

2.2. Relations between functions f_{mn}, g_{mn} and crack-face weight functions

Let $k_{mi}(a; z^{\pm}; x', z')$ ($m = 1,2,3$, $i = x,y,z$) denote the mth SIF generated at the point $z^{\pm}$ of the front of a tunnel-crack of width $2a$ by unit point forces $\pm\vec{e}_i$ exerted on the points $(x', 0^{\pm}, z')$ of the crack faces. Leblond et al. (1999) have shown that the fundamental kernel $\mathbf{Z}$ is linked to these *crack-face weight functions* by the following formula:

$$Z_{mn}(a; z^{\pm}, z'^{\pm}) = \Lambda_{ni}(z'^{\pm}) k_{mi}(a; z^{\pm}; z'^{\pm}) \tag{7}$$

where Einstein's implicit summation convention is employed for the index i. In this equation,

$$k_{mi}(a; z^{\pm}; z'^{+}) \equiv \lim_{x' \to a} \frac{k_{mi}(a; z^{\pm}; x', z')}{\sqrt{a-x'}}, \quad k_{mi}(a; z^{\pm}; z'^{-}) \equiv \lim_{x' \to -a} \frac{k_{mi}(a; z^{\pm}; x', z')}{\sqrt{a+x'}} \tag{8}$$

and the coefficients $\Lambda_{ni}(z'^{\pm})$ depend on the orientation of the local set of axes chosen to define the SIF (see Leblond et al., 1999). One verifies that for the choice made in Part I (set of axes (x,y,z) for the line $(x=a)$ and set of axes $(-x,-y,z)$ for the line $(x=-a)$),

$$\begin{aligned} \Lambda_{1y}(z'^{+}) = \Lambda_{2x}(z'^{+}) = \Lambda_{3z}(z'^{+}) = \frac{\sqrt{2\pi}}{4}, \\ \Lambda_{1y}(z'^{-}) = \Lambda_{2x}(z'^{-}) = -\Lambda_{3z}(z'^{-}) = \frac{\sqrt{2\pi}}{4}, \end{aligned} \tag{9}$$

other components being zero.

Combination of Eqs. (1), (2), (7) and (9) then yields the following formulae relating the components of $\mathbf{f}$ and $\mathbf{g}$ and the crack-face weight functions:

$$\begin{aligned}
&\frac{4}{\sqrt{2\pi}}f_{11}(z/a) = z^2 k_{1y}(a;z^+;0^+) = z^2 k_{1y}(a;z^-;0^-)\\
&\frac{4}{\sqrt{2\pi}}f_{22}(z/a) = z^2 k_{2x}(a;z^+;0^+) = z^2 k_{2x}(a;z^-;0^-)\\
&\frac{4}{\sqrt{2\pi}}f_{33}(z/a) = z^2 k_{3z}(a;z^+;0^+) = -z^2 k_{3z}(a;z^-;0^-)\\
&\frac{4}{\sqrt{2\pi}}f_{23}(z/a) = -z^2 k_{2z}(a;z^+;0^+) = z^2 k_{2z}(a;z^-;0^-) = \frac{z^2}{1-\nu}k_{3x}(a;z^+;0^+) = \frac{z^2}{1-\nu}k_{3x}(a;z^-;0^-)
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\frac{4}{\sqrt{2\pi}}g_{11}(z/a) = a^2 k_{1y}(a;z^+;0^-) = a^2 k_{1y}(a;z^-;0^+)\\
&\frac{4}{\sqrt{2\pi}}g_{22}(z/a) = a^2 k_{2x}(a;z^+;0^-) = a^2 k_{2x}(a;z^-;0^+)\\
&\frac{4}{\sqrt{2\pi}}g_{33}(z/a) = -a^2 k_{3z}(a;z^+;0^-) = a^2 k_{3z}(a;z^-;0^+)\\
&\frac{4}{\sqrt{2\pi}}g_{23}(z/a) = a^2 k_{2z}(a;z^+;0^-) = -a^2 k_{2z}(a;z^-;0^+) = \frac{a^2}{1-\nu}k_{3x}(a;z^+;0^-) = \frac{a^2}{1-\nu}k_{3x}(a;z^-;0^+)
\end{aligned} \tag{11}$$

(where parity properties of the f_{mn} and g_{mn} have been used).

3. Integrodifferential equations on the functions f_{22}, f_{33}, f_{23}, g_{22}, g_{33}, g_{23}

3.1. Overview of the method

Let us consider a tunnel-crack of width $2a$, assuming $a = 1$ without any loss of generality, subjected to a pair of unit point forces $\pm\vec{e}_i$ exerted on the crack faces. Then the SIF K_n before any perturbation of the crack front are the weight functions k_{ni}. If now the perturbation consists of a translation of one part of the front or a rotation of both parts, the crack shape is preserved so that the SIF after perturbation are also linked to the weight functions. Eq. (5) then yields equations on the weight functions. Applying the forces close to the front, one thus obtains equations on the functions $k_{mi}(a;z^{\pm};z'^{\pm})$ defined by (8). By using relations (10) and (11) connecting the functions $k_{mi}(a;z^{\pm};z'^{\pm})$ and the operators **f** and **g**, one finally obtains six integrodifferential equations on the six unknown functions f_{22}, f_{33}, f_{23}, g_{22}, g_{33}, g_{23}.

In practice, the point forces will be applied close to the point $(1,0,0)$ of the fore part of front. The index i will be taken as x or z since the choice $i = y$ would yield equations on the already known functions f_{11} and g_{11}. Two motions of the crack front will be studied:

- a translatory motion of the sole rear part of the front, the variations δK_m of the SIF being observed at the point z^+ of the fore part of the front (Fig. 1(a));
- an in-plane rotation, by an angle $\varepsilon \ll 1$, of the fore part of the front around the point $(1,0,0)$ and of the rear part of the front around the point $(-1,0,z)$, the variations δK_m of the SIF being then observed at the point z^- of the rear part of the front (Fig. 1(b)).

3.2. Equations on the weight functions

Let us consider the unperturbed crack and suppose that some unit point forces $\pm\vec{e}_i$, $i = x$ or z are applied at points $(x, 0^{\pm}, 0)$ of the crack faces. Then the SIF before any perturbation of the crack front are given by:

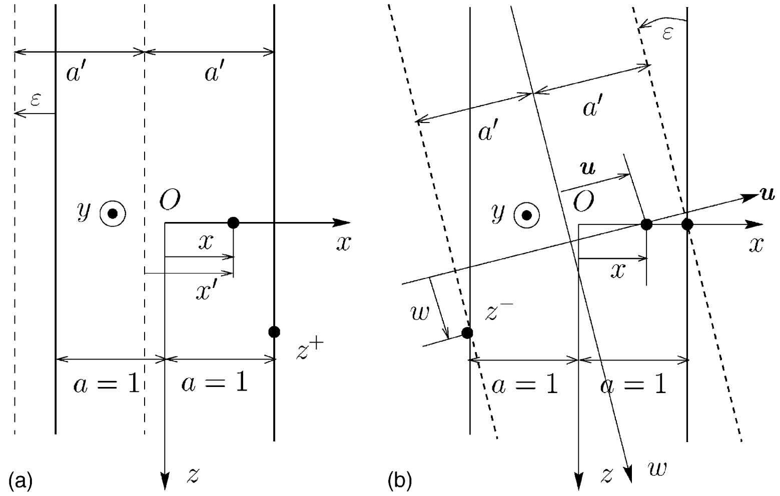

Fig. 1. Special motions of the crack front.

$$K_n(z'^{\pm}) = k_{ni}(1; z'^{\pm}; x, 0) \tag{12}$$

Let us now consider (Fig. 1(a)) a simple translatory motion of the sole rear part of the front, defined by $\delta a(z'^{+}) \equiv 0$, $\delta a(z'^{-}) \equiv \varepsilon$ where ε denotes a small parameter. Then the new SIF are those of a tunnel-crack of width $2a'$ with $a' = 1 + \varepsilon/2$, subjected to point forces exerted at the points $(x', 0^{\pm}, 0)$ with $x' = x + \varepsilon/2$ (see Fig. 1(a)). Thus

$$\delta K_m(z^+) = k_{mi}(1 + \varepsilon/2; z^+; x + \varepsilon/2, 0) - k_{mi}(1; z^+; x, 0) \tag{13}$$

so that, by Eq. (5):

$$k_{mi}(1 + \varepsilon/2; z^+; x + \varepsilon/2, 0) - k_{mi}(1; z^+; x, 0) = \varepsilon \int_{-\infty}^{+\infty} g_{mn}(z' - z) k_{ni}(1; z'^{-}; x, 0)\, \mathrm{d}z' \tag{14}$$

Next consider a rotation of the fore and rear parts of the front, defined by $\delta a(z'^{+}) \equiv \varepsilon z'$, $\delta a(z'^{-}) \equiv \varepsilon(z - z')$ where ε again denotes a small parameter (Fig. 1(b)). The axes adapted to the new front are (u, y, w) (see Fig. 1(b)). Since

$$\vec{e_x} = \vec{e_u} + \varepsilon \vec{e_w}, \quad \vec{e_z} = \vec{e_w} - \varepsilon \vec{e_u} \tag{15}$$

the SIF after perturbation at point z^- are $k_{mx}(a'; w^-; u, 0) + \varepsilon k_{mz}(a'; w^-; u, 0)$ for $i = x$ and $k_{mz}(a'; w^-; u, 0) - \varepsilon k_{mx}(a'; w^-; u, 0)$ for $i = z$, where a' again denotes the new half-width of the crack. It is easy to show that

$$a' = 1 + \frac{z}{2}\varepsilon, \quad u = x + \frac{z}{2}\varepsilon, \quad w = z - (1 + x)\varepsilon \tag{16}$$

Hence, Eq. (5) applied at point z^- yields for $m = 2$, 3: for $i = x$,

$$\begin{aligned} & k_{mx}\left(1 + \frac{z}{2}\varepsilon; [z - (1 + x)\varepsilon]^-; x + \frac{z}{2}\varepsilon, 0\right) - k_{mx}(1; z^-; x, 0) + \varepsilon k_{mz}(1; z^-; x, 0) = -\varepsilon N_{mn} k_{nx}(1; z^-; x, 0) \\ & \quad + \varepsilon \mathrm{PV} \int_{-\infty}^{+\infty} f_{mn}(z' - z) k_{nx}(1; z'^{-}; x, 0) \frac{\mathrm{d}z'}{z - z'} + \varepsilon \int_{-\infty}^{+\infty} g_{mn}(z' - z) k_{nx}(1; z'^{+}; x, 0) z'\, \mathrm{d}z' \end{aligned} \tag{17}$$

and for $i = z$:

$$k_{mz}\left(1+\frac{z}{2}\varepsilon;[z-(1+x)\varepsilon]^-;x+\frac{z}{2}\varepsilon,0\right)-k_{mz}(1;z^-;x,0)-\varepsilon k_{mx}(1;z^-;x,0)=-\varepsilon N_{mn}k_{nz}(1;z^-;x,0)$$
$$+\varepsilon\,\mathrm{PV}\int_{-\infty}^{+\infty}f_{mn}(z'-z)k_{nz}(1;z'^-;x,0)\frac{\mathrm{d}z'}{z-z'}+\varepsilon\int_{-\infty}^{+\infty}g_{mn}(z'-z)k_{nz}(1;z'^+;x,0)z'\,\mathrm{d}z' \tag{18}$$

3.3. Equations on the functions $k_{mi}(a;z^\pm;z'^\pm)$, $m = 2,3$, $i = x,z$

These equations are obtained by dividing Eqs. (14), (17), (18) by $\sqrt{1-x}$ and then taking the limit $x \to 1$, using the definition (8) of the $k_{mi}(a;z^\pm;z'^\pm)$.

For $m = 2$, 3 and $i = x$, z, one obtains:

$$\lim_{x\to1}\int_{-\infty}^{+\infty}g_{mn}(z'-z)\frac{k_{ni}(1;z'^-;x,0)}{\sqrt{1-x}}\mathrm{d}z'=\int_{-\infty}^{+\infty}g_{mn}(z'-z)k_{ni}(1;z'^-;0^+)\mathrm{d}z' \tag{19}$$

$$\lim_{x\to1}\mathrm{PV}\int_{-\infty}^{+\infty}f_{mn}(z'-z)\frac{k_{ni}(1;z'^-;x,0)}{\sqrt{1-x}}\frac{\mathrm{d}z'}{z-z'}=\mathrm{PV}\int_{-\infty}^{+\infty}f_{mn}(z'-z)k_{ni}(1;z'^-;0^+)\frac{\mathrm{d}z'}{z-z'} \tag{20}$$

These equations mean that the symbols $\lim_{x\to1}$ and $(\mathrm{PV})\int_{-\infty}^{+\infty}$ simply commute. This is because when $x \to 1$, the points $(x,0^\pm,0)$ of application of the forces do not approach the point of observation $M^-(z'^-)$ of the SIF $k_{ni}(1;z'^-;x,0)$ so that these SIF remain bounded for all z'.

However, when $x \to 1$, the points $(x,0^\pm,0)$ of application of the forces do approach the point of observation $M^+(z'^+)$ of the SIF $k_{ni}(1;z'^+;x,0)$ for the special value $z' = 0$. Thinks then become more intricate. It is thus shown in Appendix A that

$$\lim_{x\to1}\int_{-\infty}^{+\infty}g_{m2}(z'-z)\frac{k_{2x}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z'=\mathrm{PV}\int_{-\infty}^{+\infty}g_{m2}(z'-z)k_{2x}(1;z'^+;0^+)z'\,\mathrm{d}z', \tag{21}$$

a similar result holding with the substitutions $g_{m2} \to g_{m3}$, $k_{2x} \to k_{3z}$; also,

$$\lim_{x\to1}\int_{-\infty}^{+\infty}g_{m2}(z'-z)\frac{k_{2z}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z'=\mathrm{PV}\int_{-\infty}^{+\infty}g_{m2}(z'-z)k_{2z}(1;z'^+;0^+)z'\,\mathrm{d}z'+\frac{4}{\sqrt{2\pi}}\frac{v}{2-v}g_{m2}(-z), \tag{22}$$

and similarly with the substitutions $g_{m2} \to g_{m3}$, $k_{2z} \to k_{3x}$.

Combination of Eqs. (14), (17), (18) and (19)–(22) then yields the following equations on the functions $k_{mi}(a;z^\pm;z'^\pm)$, for $m = 2,3$:

$$k_{mi}(1+\varepsilon/2;z^+;0^+)-k_{mi}(1;z^+;0^+)=\varepsilon\int_{-\infty}^{+\infty}g_{mn}(z'-z)k_{ni}(1;z'^-;0^+)\,\mathrm{d}z' \tag{23}$$

for $i = x,z$, and

$$k_{mx}\left(1+\frac{z}{2}\varepsilon;[z-2\varepsilon]^-;0^+\right)-k_{mx}(1;z^-;0^+)+\varepsilon k_{mz}(1;z^-;0^+)$$
$$=-\varepsilon N_{mn}k_{nx}(1;z^-;0^+)+\varepsilon\frac{4}{\sqrt{2\pi}}\frac{v}{2-v}g_{m3}(-z)+\varepsilon\,\mathrm{PV}\int_{-\infty}^{+\infty}f_{mn}(z'-z)k_{nx}(1;z'^-;0^+)\frac{\mathrm{d}z'}{z-z'}$$
$$+\varepsilon\mathrm{PV}\int_{-\infty}^{+\infty}g_{mn}(z'-z)k_{nx}(1;z'^+;0^+)z'\,\mathrm{d}z' \tag{24}$$

$$k_{mz}\left(1+\frac{z}{2}\varepsilon;[z-2\varepsilon]^{-};0^{+}\right)-k_{mz}(1;z^{-};0^{+})-\varepsilon k_{mx}(1;z^{-};0^{+})$$
$$=-\varepsilon N_{mn}k_{nz}(1;z^{-};0^{+})+\varepsilon\frac{4}{\sqrt{2\pi}}\frac{v}{2-v}g_{m2}(-z)+\varepsilon\,\mathrm{PV}\int_{-\infty}^{+\infty}f_{mn}(z'-z)k_{nz}(1;z'^{-};0^{+})\frac{\mathrm{d}z'}{z-z'}$$
$$+\varepsilon\,\mathrm{PV}\int_{-\infty}^{+\infty}g_{mn}(z'-z)k_{nz}(1;z'^{+};0^{+})z'\,\mathrm{d}z' \tag{25}$$

3.4. Equations on the functions f_{22}, f_{33}, f_{23}, g_{22}, g_{33}, g_{23}

Using relations (10) and (11) and identifying terms of order ε in Eqs. (23)–(25), one obtains the following integrodifferential equations on the functions:

$$f_{22}'(z)=-2z\int_{-\infty}^{+\infty}[g_{22}(z-z')g_{22}(z')-(1-v)g_{23}(z-z')g_{23}(z')]\,\mathrm{d}z' \tag{26}$$

$$f_{33}'(z)=-2z\int_{-\infty}^{+\infty}[g_{33}(z-z')g_{33}(z')-(1-v)g_{23}(z-z')g_{23}(z')]\,\mathrm{d}z' \tag{27}$$

$$f_{23}'(z)=-2z\int_{-\infty}^{+\infty}g_{23}(z-z')(g_{22}+g_{33})(z')\,\mathrm{d}z' \tag{28}$$

$$\left[\left(1+\frac{z^2}{4}\right)g_{22}(z)\right]'+\frac{2(1-v)}{2-v}g_{23}(z)=(1-v)\int_{-\infty}^{+\infty}g_{23}(z-z')\frac{f_{23}(z')}{z'}\mathrm{d}z'$$
$$-\mathrm{PV}\int_{-\infty}^{+\infty}g_{22}(z-z')\frac{f_{22}(z')}{z'}\mathrm{d}z' \tag{29}$$

$$\left[\left(1+\frac{z^2}{4}\right)g_{33}(z)\right]'+\frac{2(1-v)}{2-v}g_{23}(z)=(1-v)\int_{-\infty}^{+\infty}g_{23}(z-z')\frac{f_{23}(z')}{z'}\mathrm{d}z'$$
$$-\mathrm{PV}\int_{-\infty}^{+\infty}g_{33}(z-z')\frac{f_{33}(z')}{z'}\mathrm{d}z' \tag{30}$$

$$\left[\left(1+\frac{z^2}{4}\right)g_{23}(z)\right]'-\frac{1}{2-v}(g_{22}+g_{33})(z)=-\int_{-\infty}^{+\infty}(g_{22}+g_{33})(z-z')\frac{f_{23}(z')}{2z'}\mathrm{d}z'$$
$$-\mathrm{PV}\int_{-\infty}^{+\infty}g_{23}(z-z')\frac{(f_{22}+f_{33})(z')}{2z'}\mathrm{d}z' \tag{31}$$

where use has been made of relations (6) and parity properties of the f_{mn} and g_{mn} (see Section 2.1).

4. Differential equations and initial conditions on the functions $\widehat{F}_{22}$, $\widehat{F}_{33}$, $\widehat{F}_{23}$, $\hat{g}_{22}$, $\hat{g}_{33}$, $\hat{g}_{23}$

The definition adopted for the Fourier transform $\hat{\varphi}(p)$ of some function $\varphi(z)$ is the same as in Part I:

$$\hat{\varphi}(p)\equiv\int_{-\infty}^{+\infty}\varphi(z)\mathrm{e}^{\mathrm{i}pz}\,\mathrm{d}z \tag{32}$$

Note that since f_{22}, f_{33}, g_{22}, g_{33} are even and f_{23}, g_{23} odd, $\hat{f}_{22}$, $\hat{f}_{33}$, $\hat{g}_{22}$, $\hat{g}_{33}$ are even and real, and $\hat{f}_{23}$, $\hat{g}_{23}$ odd and purely imaginary.

4.1. Differential equations

Taking the Fourier transform of Eqs. (26)–(31) is elementary except for terms of the form

$$(\mathrm{PV})\int_{-\infty}^{+\infty} g_{mn}(z-z')\frac{f_{rs}(z')}{z'}\,\mathrm{d}z',$$

which are envisaged in Appendix B. The resulting equations read as follows:

$$\widehat{F}'_{22} = -\tfrac{2}{p}[\hat{g}_{22}^2 - (1-\nu)\hat{g}_{23}^2]' \tag{33}$$

$$\widehat{F}'_{33} = -\tfrac{2}{p}[\hat{g}_{33}^2 - (1-\nu)\hat{g}_{23}^2]' \tag{34}$$

$$\widehat{F}'_{23} = -\tfrac{2}{p}[\hat{g}_{23}(\hat{g}_{22}+\hat{g}_{33})]' \tag{35}$$

$$\hat{g}_{22} - \frac{\hat{g}''_{22}}{4} = \tfrac{1}{p}[\widehat{F}_{22}\hat{g}_{22} - (1-\nu)\widehat{F}_{23}\hat{g}_{23}] \tag{36}$$

$$\hat{g}_{33} - \frac{\hat{g}''_{33}}{4} = \tfrac{1}{p}[\widehat{F}_{33}\hat{g}_{33} - (1-\nu)\widehat{F}_{23}\hat{g}_{23}] \tag{37}$$

$$\hat{g}_{23} - \frac{\hat{g}''_{23}}{4} = \frac{1}{p}\left[\frac{\widehat{F}_{22}+\widehat{F}_{33}}{2}\hat{g}_{23} + \widehat{F}_{23}\frac{\hat{g}_{22}+\hat{g}_{33}}{2}\right] \tag{38}$$

In these expressions, the functions $\widehat{F}_{mn}$ are the definite integrals of the functions $\hat{f}_{mn}$ defined by

$$\widehat{F}_{mn}(p) \equiv \int_0^p \hat{f}_{mn}(q)\,\mathrm{d}q + \begin{cases} 0, & (m,n)=(2,2),(3,3) \\ \mathrm{i}\,\frac{2-\nu}{2(1-\nu)}, & (m,n)=(2,3) \end{cases} \tag{39}$$

Note that since $\hat{f}_{22}$, $\hat{f}_{33}$ are even and $\hat{f}_{23}$ odd, $\widehat{F}_{22}$, $\widehat{F}_{33}$ are odd and $\widehat{F}_{23}$ even. Because of these parity properties and those of the $\hat{g}_{mn}$, it suffices to determine all functions on the interval $(0,+\infty)$. Also, note that functions $\widehat{F}_{22}$, $\widehat{F}_{33}$ are real and $\widehat{F}_{23}$ purely imaginary.

Eqs. (33)–(38) form a system of six non-linear differential equations (on the interval $(0,+\infty)$), on the six unknown functions $\widehat{F}_{22}$, $\widehat{F}_{33}$, $\widehat{F}_{23}$, $\hat{g}_{22}$, $\hat{g}_{33}$, $\hat{g}_{23}$, of order 1 with respect to the $\widehat{F}_{mn}$ and order 2 with respect to the $\hat{g}_{mn}$. Hence, to (numerically) get these functions on any interval $[p_0, p_\infty]$ with $0 < p_0 \ll 1$ and $p_\infty \gg 1$, one may proceed in two ways:

- integrate "forwards", from p_0 to p_∞; this requires knowing the values of the $\widehat{F}_{mn}$, $\hat{g}_{mn}$ and $\hat{g}'_{mn}$ at p_0, that is near 0;
- integrate "backwards", from p_∞ to p_0; the values of the $\widehat{F}_{mn}$, $\hat{g}_{mn}$ and $\hat{g}'_{mn}$ are then needed at p_∞, that is near $+\infty$.

The next sections are therefore devoted to the necessary asymptotic study of the functions near 0 and $+\infty$.

4.2. Values of $\widehat{F}_{22}(0)$, $\widehat{F}_{33}(0)$, $\widehat{F}_{23}(0)$, $\hat{g}_{22}(0)$, $\hat{g}_{33}(0)$, $\hat{g}_{23}(0)$

From the definition (39) of the $\widehat{F}_{mn}$, it is clear that:

$$\widehat{F}_{22}(0) = \widehat{F}_{33}(0) = 0, \quad \widehat{F}_{23}(0) = \mathrm{i}\frac{2-v}{2(1-v)} \tag{40}$$

Moreover, values of $\hat{g}_{22}(0)$, $\hat{g}_{33}(0)$, $\hat{g}_{23}(0)$ are given by relations (14) of Part I, recalled here for the sake of completeness:

$$\hat{g}_{22}(0) = -\hat{g}_{33}(0) = \frac{1}{4}, \quad \hat{g}_{23}(0) = 0 \tag{41}$$

Finally, the derivatives $\hat{g}'_{mn}(0)$ are given by $\hat{g}'_{mn}(0) = i\int_{-\infty}^{+\infty} z g_{mn}(z)\,\mathrm{d}z$. For $(m,n) = (2,3)$, this integral is given by Eq. (18) of Part I. For $(m,n) = (2,2), (3,3)$, it is zero since $\hat{g}_{mn}$ is even. In conclusion,

$$\hat{g}'_{22}(0) = \hat{g}'_{33}(0) = 0, \quad \hat{g}'_{23}(0) = \mathrm{i}\frac{v}{2(1-v)} \tag{42}$$

4.3. Asymptotic behavior of $\widehat{F}_{22}(p)$, $\widehat{F}_{33}(p)$, $\widehat{F}_{23}(p)$, $\hat{g}_{22}(p)$, $\hat{g}_{33}(p)$, $\hat{g}_{23}(p)$ for $p \to 0^+$

As a first approximation, the values of the $\widehat{F}_{mn}(p_0)$, $\hat{g}_{mn}(p_0)$ and $\hat{g}'_{mn}(p_0)$ may be taken equal to those of the $\widehat{F}_{mn}(0)$, $\hat{g}_{mn}(0)$ and $\hat{g}'_{mn}(0)$. However, more refined values can be found by studying the asymptotic behavior of the functions near 0. Such a study will also be needed to derive asymptotic formulae for the $f_{mn}(z)$ and $g_{mn}(z)$ for $z \to +\infty$, which will nicely supplement the numerical values found over some necessary finite interval.

Let us suppose that just as the function $\hat{g}_{11}(p)$ (Leblond et al., 1996), $\hat{g}_{22}(p)$, $\hat{g}_{33}(p)$, $\hat{g}_{23}(p)$ admit, for $p \to 0^+$, a development involving terms of the form $p^\alpha \ln^\beta p$, α, $\beta \in \mathbb{N}$. With this hypothesis, it is shown in Appendix C that:

$$\hat{g}_{22}(p) = \frac{1}{4} + \frac{1-2v}{4}p^2 \ln p + \mathrm{O}(p^2) \tag{43}$$

$$\hat{g}_{33}(p) = -\frac{1}{4} - \frac{1+v}{4(1-v)}p^2 \ln p + \mathrm{O}(p^2) \tag{44}$$

$$\hat{g}_{23}(p) = \mathrm{i}\frac{v}{2(1-v)}p + \mathrm{i}\frac{v(v^2-2v+2)}{4(1-v)^2}p^3 \ln p + \mathrm{O}(p^3) \tag{45}$$

$$\widehat{F}_{22}(p) = -\frac{1-2v}{2}p \ln p + \mathrm{O}(p) \tag{46}$$

$$\widehat{F}_{33}(p) = -\frac{1+v}{2(1-v)}p \ln p + \mathrm{O}(p) \tag{47}$$

$$\widehat{F}_{23}(p) = \mathrm{i}\frac{2-v}{2(1-v)} + \mathrm{i}\frac{3v^2(2-v)}{4(1-v)^2}p^2 \ln p + \mathrm{O}(p^2) \tag{48}$$

for $p \to 0^+$.

4.4. Asymptotic behavior of $\widehat{F}_{22}(p)$, $\widehat{F}_{33}(p)$, $\widehat{F}_{23}(p)$, $\hat{g}_{22}(p)$, $\hat{g}_{33}(p)$, $\hat{g}_{23}(p)$ for $p \to +\infty$

For $p \to +\infty$, the set of Eqs. (33)–(38) approximately reads as follows:

$$\widehat{F}'_{mn} = 0, \quad \hat{g}_{mn} - \frac{\hat{g}''_{mn}}{4} = 0, \quad (m,n) = (2,2), (3,3), (2,3) \tag{49}$$

Eq. $(49)_1$ strongly suggests that the $\widehat{F}_{mn}(p)$ tend toward some finite limits for $p \to +\infty$. The determination of these limits, noted $\widehat{F}^{\infty}_{mn}$, is expounded in Appendix D. The results are as follows:

$$\widehat{F}^{\infty}_{22} = \frac{2-3\nu}{2(2-\nu)}, \quad \widehat{F}^{\infty}_{33} = \frac{2+\nu}{2(2-\nu)}, \quad \widehat{F}^{\infty}_{23} = \mathrm{i}\frac{2}{2-\nu} \tag{50}$$

Eq. $(49)_2$ shows that the $\hat{g}_{mn}(p)$ behave like $\mathrm{e}^{\pm 2p}$ for $p \to +\infty$. However, the increasing component e^{2p} is obviously physically inadmissible, so that the $\hat{g}_{mn}(p)$ must behave like e^{-2p}. Unfortunately, Eq. $(49)_2$ fails to provide the values of the pre-exponential factors here.

5. Determination of functions f_{22}, f_{33}, f_{23}, g_{22}, g_{33}, g_{23}

Prior to giving numerical results, we briefly discuss here how the functions f_{mn} and g_{mn} can be obtained from the $\widehat{F}_{mn}$ and $\hat{g}_{mn}$.

5.1. Inverse Fourier transform

The g_{mn} are readily deduced from the $\hat{g}_{mn}$ through Fourier inversion:

$$g_{mn}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}_{mn}(p)\mathrm{e}^{-\mathrm{i}pz}\,\mathrm{d}p = \frac{1}{\pi}\int_{0}^{+\infty} \hat{g}_{mn}(p)\cos pz\,\mathrm{d}p, \quad (m,n) = (2,2), (3,3) \tag{51}$$

$$g_{23}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}_{23}(p)\mathrm{e}^{-\mathrm{i}pz}\,\mathrm{d}p = -\frac{1}{\pi}\int_{0}^{+\infty} i\hat{g}_{23}(p)\sin pz\,\mathrm{d}p \tag{52}$$

where parity properties of the $\hat{g}_{mn}$ have been used.

Also, $\hat{f}_{mn} = \widehat{F}'_{mn}$, so that:

$$f_{mn}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{F}'_{mn}(p)\mathrm{e}^{-\mathrm{i}pz}\,\mathrm{d}p = \frac{1}{\pi}\int_{0}^{+\infty} \widehat{F}'_{mn}(p)\cos pz\,\mathrm{d}p, \quad (m,n) = (2,2), (3,3) \tag{53}$$

$$f_{23}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{F}'_{23}(p)\mathrm{e}^{-\mathrm{i}pz}\,\mathrm{d}p = -\frac{1}{\pi}\int_{0}^{+\infty} i\widehat{F}'_{23}(p)\sin pz\,\mathrm{d}p \tag{54}$$

where parity properties have again been used.

The functions f_{mn} and g_{mn} will be determined numerically from these formulae over some finite interval, say for $0 \leqslant z \leqslant z_\infty$, $z_\infty \gg 1$. For values of $z > z_\infty$, one may use the asymptotic expressions given below.

5.2. Asymptotic behavior of f_{22}, f_{33}, f_{23}, g_{22}, g_{33}, g_{23} for $z \to +\infty$

The derivation of these behaviors is a little complex and hence relegated to Appendix E. Only the final results are given below:

$$g_{22}(z) \sim \frac{1-2v}{4z^3}, \quad g_{33}(z) \sim -\frac{1+v}{4(1-v)z^3}, \quad g_{23}(z) \sim \frac{3v(v^2-2v+2)}{4(1-v)^2 z^4} \tag{55}$$

$$f_{22}(z) \sim \frac{1-2v}{4z}, \quad f_{33}(z) \sim \frac{1+v}{4(1-v)z}, \quad f_{23}(z) \sim -\frac{3v^2(2-v)}{4(1-v)^2 z^2} \tag{56}$$

6. Numerical procedure and results

6.1. Calculation of $\widehat{F}_{22}$, $\widehat{F}_{33}$, $\widehat{F}_{23}$, $\hat{g}_{22}$, $\hat{g}_{33}$, $\hat{g}_{23}$

As mentioned above, the set of differential equations (33)–(38) can be solved on any interval $[p_0, p_\infty]$ with $0 < p_0 \ll 1$ and $p_\infty \gg 1$, by integrating "forwards", from p_0 to p_∞, or "backwards", from p_∞ to p_0. Let us compare these two methods:

- Integrating "forwards" seems, a priori, more suitable since the values of the $\widehat{F}_{mn}(p_0)$, $\hat{g}_{mn}(p_0)$, $\hat{g}'_{mn}(p_0)$ are known (Eqs. (43)–(48)), in contrast to the precise asymptotic behavior of the $\hat{g}_{mn}(p)$ and $\hat{g}'_{mn}(p)$ for $p \to +\infty$. However, due to the behavior in $e^{\pm 2p}$ of the $\hat{g}_{mn}(p)$ at infinity, any (inevitable) numerical error in the initial conditions or the integration method will yield a spurious component in e^{2p} in the $\hat{g}_{mn}(p)$ that will quickly "blow up", thus prohibiting to reach large values of p.
- Hence the only possibility is to integrate "backwards". The values of the $\widehat{F}_{mn}(p_\infty)$, $\hat{g}_{mn}(p_\infty)$, $\hat{g}'_{mn}(p_\infty)$ are then needed, but only these of the $\widehat{F}_{mn}(p_\infty)$ are known (Eq. (50)). To determine those of the $\hat{g}_{mn}(p_\infty)$ and $\hat{g}'_{mn}(p_\infty)$, one can use a Newton method aimed at matching the values of the $\hat{g}_{mn}(p_0)$ and $\hat{g}'_{mn}(p_0)$ given by Eqs. (43)–(45). (One can show that the values obtained for the $\widehat{F}_{mn}(p_0)$ necessarily then match conditions (46)–(48).) This task is not straightforward because one must first find good "initial values" for the $\hat{g}_{mn}(p_\infty)$ and $\hat{g}'_{mn}(p_\infty)$ in the Newton method, ensuring convergence of the algorithm. Indeed, for many choices of these initial values, the functions diverge toward infinity when p approaches p_0, due to the singularity in $1/p$ of the differential equations.

In practice, the Runge–Kutta method of order four is used to integrate from $p_\infty = 50$ to $p_0 = 10^{-6}$ with an accuracy of 10^{-5}. The solutions obtained for $v = 0.1$ and $v = 0.3$ are given in Figs. 2–5.

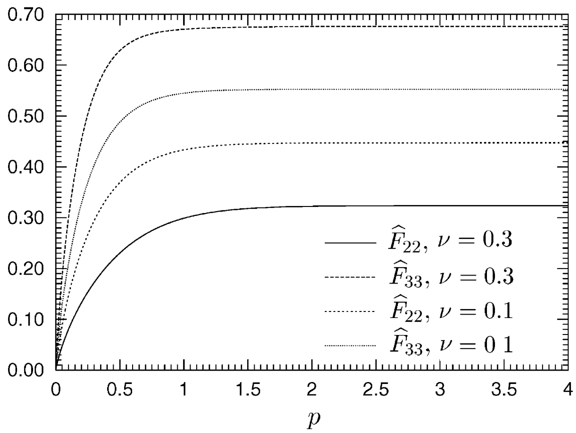

Fig. 2. Functions $\widehat{F}_{22}(p)$, $\widehat{F}_{33}(p)$.

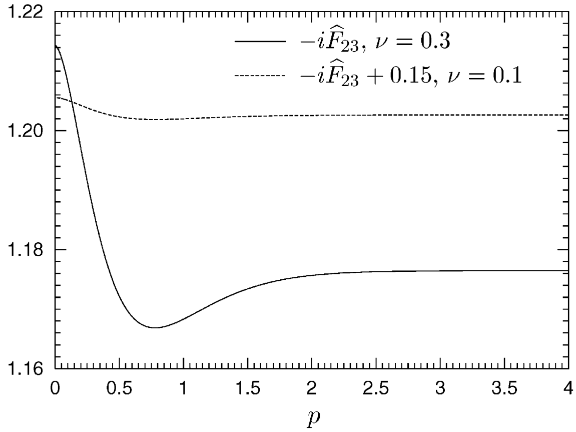

Fig. 3. Function $\widehat{F}_{23}(p)$.

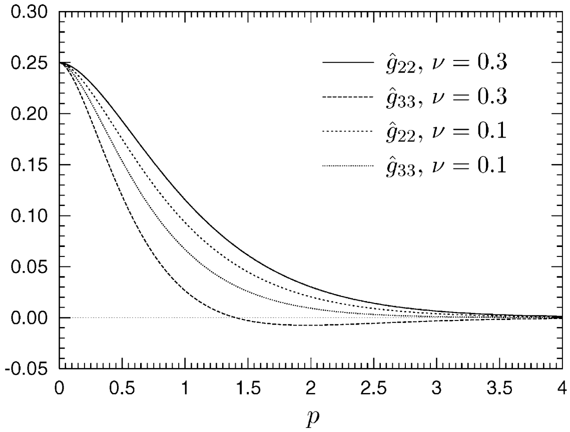

Fig. 4. Functions $\hat{g}_{22}(p)$, $\hat{g}_{33}(p)$.

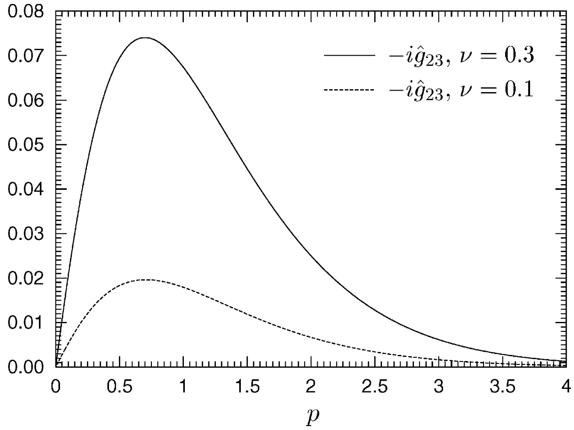

Fig. 5. Function $\hat{g}_{23}(p)$.

6.2. Calculation of $\bar{f}_{22}$, $\bar{f}_{33}$

Functions $\bar{f}_{22}$, $\bar{f}_{33}$ (in addition to $\hat{g}_{22}$, $\hat{g}_{33}$ and $\hat{g}_{23}$) were needed in Part I for the study of the bifurcation and stability problems. They are defined by Eq. (23) of Part I. One can easily show that $\bar{f}_{mn}(0) = 1/4$, $\bar{f}'_{mn} = -\widehat{F}_{mn}$, $(m,n) = (2,2),(3,3)$. Thus $\bar{f}_{22}$ and $\bar{f}_{33}$ can be obtained numerically through integration of $\widehat{F}_{22}$ and $\widehat{F}_{33}$. The results are given in Fig. 3 of Part I.

6.3. Calculation of operators *f* and *g*

It is recalled that the operators **f** and **g** are linked to the fundamental kernel **Z** by relations (1) and (2), that their components 11 are given in Leblond et al. (1996) and that their components 12, 21, 13, 31 are zero. The other components are obtained by using Eqs. (51)–(54) and (4). In practice, the integration interval $[0, +\infty)$ is replaced by the interval $[10^{-6}, 50]$, and calculations are performed for $z \in [0, 50]$. Functions f_{22}, f_{33}, f_{23} are presented in Figs. 6 and 7 for $z \in [0, 20]$, and functions g_{22}, g_{33}, g_{23} in Figs. 8 and 9 for $z \in [0, 6]$. Beyond these limits, the asymptotic expressions (55) and (56) are found to fit very well to the numerical results.

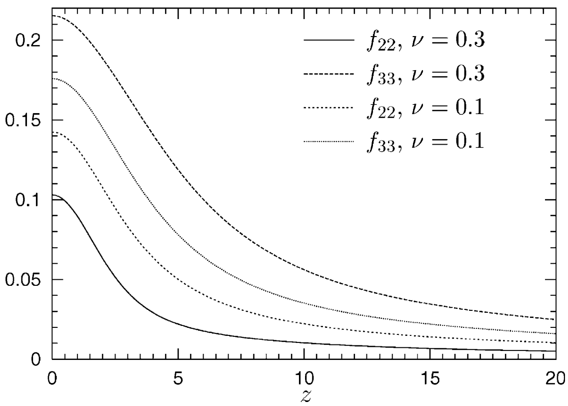

Fig. 6. Functions $f_{22}(z)$, $f_{33}(z)$.

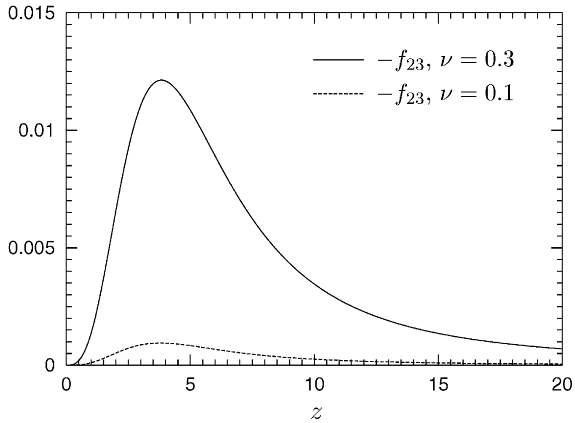

Fig. 7. Function $f_{23}(z)$.

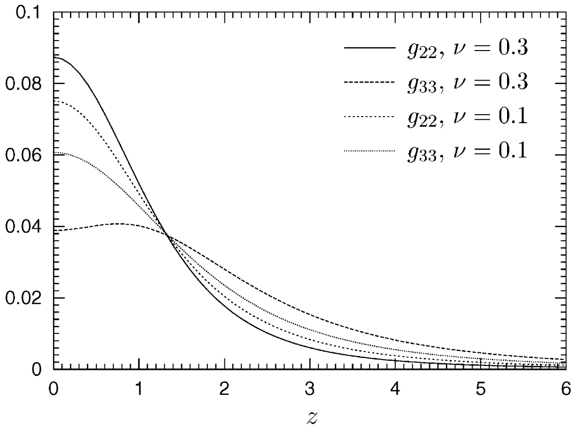

Fig. 8. Functions $g_{22}(z)$, $g_{33}(z)$.

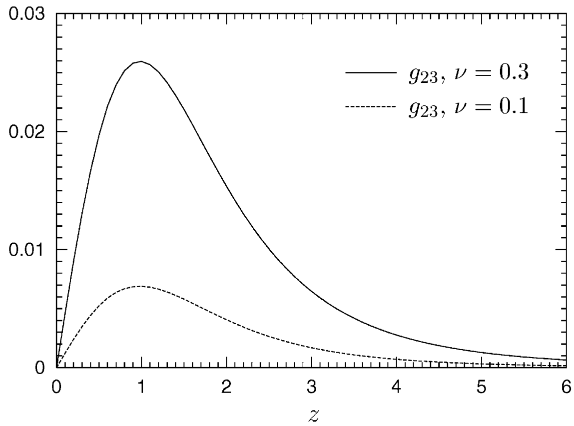

Fig. 9. Function $g_{23}(z)$.

One can observe that components 23 of operators **f** and **g**, which represent the coupling effect between modes 2 and 3, are all the smaller as Poisson's ratio ν is low. Also, it is somewhat surprising that for $\nu = 0.3$, $g_{33}(z)$, which is tied to the 3rd SIF at point z^- of the rear part of the front when point forces $\pm\vec{e}_z$ are applied close to the point 0^+ of the fore part of the front, is *not* maximum for $z = 0$. Note, however, that a similar phenomenon is known to occur for the half-plane crack: the mode 3 SIF generated by point forces $\pm\vec{e}_z$ exerted on the crack faces is not maximum at that point of the crack front located closest to the points of application of the forces. Also, this effect can be observed to vanish for sufficiently small Poisson's ratios.

Appendix A. Justification of formulae (21), (22)

To calculate $\lim_{x\to 1} \int_{-\infty}^{+\infty} g_{mn}(z'-z)((k_{ni}(1;z'^+;x,0))/(\sqrt{1-x}))z'\,\mathrm{d}z'$, split the integration domain $(-\infty,+\infty)$ into $(-\infty,-\eta)\cup(\eta,+\infty)$ and $[-\eta,\eta]$, η being a momentarily fixed arbitrary positive number. For $z' \in (-\infty,-\eta)\cup(\eta,+\infty)$, $((k_{ni}(1;z'^+;x,0))/(\sqrt{1-x}))$ has a finite limit for $x \to 1$, equal to $k_{ni}(1;z'^+;0^+)$ by definition, since the observation point z'^+ of the SIF differs from the limit-points $(x = 1, y = 0^\pm, z = 0)$ of application of the point forces. Hence

$$\lim_{x\to 1}\int_{(-\infty,-\eta)\cup(\eta,+\infty)} g_{mn}(z'-z)\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' = \int_{(-\infty,-\eta)\cup(\eta,+\infty)} g_{mn}(z'-z)k_{ni}(1;z'^+;0^+)z'\,\mathrm{d}z' \tag{A.1}$$

To evaluate the limit, for $x \to 1$, of the integral over $[-\eta,\eta]$, let us perform a first order Taylor expansion of the quantity $g_{mn}(z'-z)$ around the point $z'=0$:

$$\int_{-\eta}^{\eta} g_{mn}(z'-z)\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' = g_{mn}(-z)\int_{-\eta}^{\eta}\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' + \int_{-\eta}^{\eta}\mathrm{O}(z')\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' \tag{A.2}$$

The examples of the semi-infinite crack, the penny-shaped crack and the tunnel-crack in mode 1 strongly suggest that $(k_{ni}(1;z'^+;x,0)/\sqrt{1-x})$ is bounded by $Cst.z'^{-2}$ for $z'\to 0$. Therefore the integrand in the second term of the right-hand side of Eq. (A.2) is $\mathrm{O}(1)$, so that this equation may be rewritten as:

$$\int_{-\eta}^{\eta} g_{mn}(z'-z)\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' = g_{mn}(-z)\int_{-\eta}^{\eta}\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' + \mathrm{O}(\eta) \tag{A.3}$$

The integral in the right-hand side of (A.3) is zero if $k_{ni}(1;z'^+;x,0)$ is even with respect to z', that is for $(n,i)=(2,x),(3,z)$. On the other hand, if $k_{ni}(1;z'^+;x,0)$ is odd, i.e. for $(n,i)=(2,z),(3,x)$, one gets upon use of the homogeneity property of $k_{ni}(1;z'^+;x,0)$ and the change of variable $z''=z'/(1-x)$:

$$\int_{-\eta}^{\eta}\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' = \int_{-\eta/(1-x)}^{\eta/(1-x)} k_{ni}(1/(1-x);z''^+;x/(1-x),0)\,z''\,\mathrm{d}z''$$

The $k_{ni}(1/(1-x);z''^+;x/(1-x),0)$ are the weight functions of the tunnel-crack of width $2/(1-x)$ at point z''^+ when the forces are applied at a distance $1/(1-x)-x/(1-x)=1$ from the fore part of the front. Since when $x\to 1$, this width becomes infinity, the $k_{ni}(1/(1-x);z''^+;x/(1-x),0)$ behave as the weight functions of a half-plane crack when the forces are applied at a distance of unity from the front. Using the well-known expressions of these weight functions (see for instance Gao and Rice (1986)), one then gets for $(n,i)=(2,z),(3,x)$:

$$\lim_{x\to 1}\int_{-\eta/(1-x)}^{\eta/(1-x)} k_{ni}(1/(1-x);z''^+;x/(1-x),0)z''\,\mathrm{d}z'' = \sqrt{\frac{1}{2\pi}\frac{4\nu}{2-\nu}}$$

It follows from these elements that (A.3) finally reads, in the limit $x\to 1$:

$$\lim_{x\to 1}\int_{-\eta}^{\eta} g_{mn}(z'-z)\frac{k_{ni}(1;z'^+;x,0)}{\sqrt{1-x}}z'\,\mathrm{d}z' = \begin{cases}\mathrm{O}(\eta) & \text{if } (n,i)=(2,x) \text{ or } (3,z)\\ \sqrt{\frac{1}{2\pi}\frac{4\nu}{2-\nu}}\,g_{mn}(-z)+\mathrm{O}(\eta) & \text{if } (n,i)=(2,z) \text{ or } (3,x)\end{cases} \tag{A.4}$$

Combination of Eqs. (A.1) and (A.4), in the limit $\eta\to 0$, finally yields relations (21) and (22).

Appendix B. Calculation of some Fourier transforms

By definition, the symbol FT denoting the Fourier transform:

$$\begin{aligned}\mathrm{FT}\left[\mathrm{PV}\int_{-\infty}^{+\infty} g_{mn}(z-z')\frac{f_{rs}(z')}{z'}\mathrm{d}z'\right](p) &= \lim_{\eta\to 0}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}pz}\,\mathrm{d}z\int_{(-\infty,-\eta)\cup(\eta,+\infty)} g_{mn}(z-z')\frac{f_{rs}(z')}{z'}\mathrm{d}z'\\ (z''=z-z') &= \lim_{\eta\to 0}\int_{-\infty}^{+\infty} g_{mn}(z'')\mathrm{e}^{\mathrm{i}pz''}\,\mathrm{d}z''.\int_{(-\infty,-\eta)\cup(\eta,+\infty)}\frac{f_{rs}(z')}{z'}\mathrm{e}^{\mathrm{i}pz'}\,\mathrm{d}z'\\ &\times \hat{g}_{mn}(p)\mathrm{PV}\int_{-\infty}^{+\infty}\frac{f_{rs}(z)}{z}\mathrm{e}^{\mathrm{i}pz}\,\mathrm{d}z\end{aligned} \tag{B.1}$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{PV}\int_{-\infty}^{+\infty}\frac{f_{rs}(z)}{z}\mathrm{e}^{\mathrm{i}pz}\,\mathrm{d}z=\int_{-\infty}^{+\infty}f_{rs}(z)\mathrm{i}\mathrm{e}^{\mathrm{i}pz}\,\mathrm{d}z=\mathrm{i}\hat{f}_{rs}(p)\Rightarrow\mathrm{PV}\int_{-\infty}^{+\infty}\frac{f_{rs}(z)}{z}\mathrm{e}^{\mathrm{i}pz}\,\mathrm{d}z$$
$$=\mathrm{i}\int_0^p\hat{f}_{rs}(q)\mathrm{d}q+\mathrm{PV}\int_{-\infty}^{+\infty}\frac{f_{rs}(z)}{z}\,\mathrm{d}z \tag{B.2}$$

Now, for $(r,s)=(2,2)$ or $(3,3)$, f_{rs} is even. Eqs. (B.1) and (B.2) then yield:

$$\mathrm{FT}\left[\mathrm{PV}\int_{-\infty}^{+\infty}g_{mn}(z-z')\frac{f_{rs}(z')}{z'}\mathrm{d}z'\right](p)=i\,\hat{g}_{mn}(p)\,\widehat{F}_{rs}(p), \tag{B.3}$$

$$\widehat{F}_{rs}(p)\equiv\int_0^p\hat{f}_{rs}(q)\,\mathrm{d}q \tag{B.4}$$

for $(r,s)=(2,2),\ (3,3)$.

Similarly, for $(r,s)=(2,3)$, Eqs. (B.1) and (B.2) yield:

$$\mathrm{FT}\left[\int_{-\infty}^{+\infty}g_{mn}(z-z')\frac{f_{23}(z')}{z'}\mathrm{d}z'\right](p)=\hat{g}_{mn}(p)\left[i\int_0^p\hat{f}_{23}(q)\,\mathrm{d}q+\int_{-\infty}^{+\infty}\frac{f_{23}(z)}{z}\mathrm{d}z\right]$$

But $\int_{-\infty}^{+\infty}(f_{23}(z)/z)\,\mathrm{d}z$ is given by Eq. (17) of Part I. Thus,

$$\mathrm{FT}\left[\int_{-\infty}^{+\infty}g_{mn}(z-z')\frac{f_{23}(z')}{z'}\mathrm{d}z'\right](p)=\hat{g}_{mn}(p)\left[\mathrm{i}\widehat{F}_{23}(p)+\frac{2}{2-v}\right], \tag{B.5}$$

$$\widehat{F}_{23}(p)\equiv\int_0^p\hat{f}_{23}(q)\,\mathrm{d}q+\mathrm{i}\frac{2-v}{2(1-v)} \tag{B.6}$$

The choice of the additive constant $\mathrm{i}((2-v)/(2(1-v)))$ in the definition of $\widehat{F}_{23}$ here was made in order to simplify the differential equations (33)–(38) as much as possible.

Appendix C. Determination of the asymptotic behavior of $\widehat{F}_{22}(p)$, $\widehat{F}_{33}(p)$, $\widehat{F}_{23}(p)$, $\hat{g}_{22}(p)$, $\hat{g}_{33}(p)$, $\hat{g}_{23}(p)$ for $p\to 0^+$

It is assumed that $\hat{g}_{22}(p),\hat{g}_{33}(p),\hat{g}_{23}(p)$ admit, for $p\to 0^+$, expansions involving terms of the form $p^\alpha\ln^\beta p$ $(\alpha,\beta\in\mathbb{N})$. By Eqs. (41) and (42), these expansions read

$$\hat{g}_{22}(p)=1/4+a_2p^2\ln^{\beta_2}p+\mathrm{O}(p^2\ln^{\beta_2-1}p) \tag{C.1}$$

$$\hat{g}_{33}(p)=-1/4+a_3p^2\ln^{\beta_3}p+\mathrm{O}(p^2\ln^{\beta_3-1}p) \tag{C.2}$$

$$\hat{g}_{23}(p)=\mathrm{i}\frac{v}{2(1-v)}p+a_4p^3\ln^{\beta_4}p+\mathrm{O}(p^3\ln^{\beta_4-1}p) \tag{C.3}$$

The absence of a term of the form $p^2\ln^\beta p$ in $\hat{g}_{23}(p)$ here can be checked to be compatible with the differential equations (33)–(38).

Inserting these equations into the set of differential equations (33)–(38) and identifying principal terms, one first gets

$$\beta_2=\beta_3=\beta_4=1 \tag{C.4}$$

To next determine the constants a_2, a_3, a_4, one must consider the terms proportional to p^2 in the expansions of $\hat{g}_{22}(p)$ and $\hat{g}_{33}(p)$, since the derivatives of both expressions $p^2 \ln p$ and p^2 involve terms of the same order p. Thus, let us write these expansions in the form

$$\hat{g}_{22}(p) = 1/4 + a_2 p^2 \ln p + b_2 p^2 + \mathrm{O}(p^3 \ln^{\gamma_2} p)$$

$$\hat{g}_{33}(p) = -1/4 + a_3 p^2 \ln p + b_3 p^2 + \mathrm{O}(p^3 \ln^{\gamma_3} p)$$

Eqs. (33)–(35) then yield, account being taken of (39):

$$\widehat{F}_{22}(p) = -2a_2 p \ln p - \left(2b_2 - a_2 + \frac{v^2}{1-v}\right)p + \mathrm{O}(p^2 \ln^{\gamma_2} p) \tag{C.5}$$

$$\widehat{F}_{33}(p) = 2a_3 p \ln p + \left(2b_3 - a_3 - \frac{v^2}{1-v}\right)p + \mathrm{O}(p^2 \ln^{\gamma_3} p) \tag{C.6}$$

$$\widehat{F}_{23}(p) = \mathrm{i}\frac{2-v}{2(1-v)} - \mathrm{i}\frac{3v}{2(1-v)}(a_2 + a_3)p^2 \ln p + \mathrm{O}(p^2) \tag{C.7}$$

Inserting these expressions into (36)–(38), we finally get, after a long but straightforward calculation, by identifying terms of identical order:

$$a_2 = \frac{1-2v}{4}, \quad a_3 = -\frac{1+v}{4(1-v)}, \quad a_4 = \mathrm{i}\frac{v(v^2-2v+2)}{4(1-v)^2} \tag{C.8}$$

Although the introduction of coefficients b_2, b_3 in the reasoning was necessary for the reason explained above, they are found to finally cancel out in the calculation, which therefore fails to yield their values.

Eqs. (C.1)–(C.4) and (C.8) justify Eqs. (43)–(45) of the text, and Eqs. (C.5)–(C.8) justify Eqs. (46)–(48).

Appendix D. Determination of constants $\widehat{F}_{22}^{\infty}$, $\widehat{F}_{33}^{\infty}$, $\widehat{F}_{23}^{\infty}$

Eqs. (B.2) and (B.4) yield, since f_{22} is an even function:

$$\widehat{F}_{22}^{\infty} \equiv \lim_{p\to+\infty} \widehat{F}_{22}(p) = \lim_{p\to+\infty} 2\int_0^{+\infty} \frac{\sin pz}{z} f_{22}(z)\,\mathrm{d}z$$

$$\stackrel{pz\equiv u}{=} \lim_{p\to+\infty} 2\int_0^{+\infty} \frac{\sin u}{u} f_{22}(u/p)\,\mathrm{d}u = \pi f_{22}(0)$$

It then follows from Eq. (7) of Part I that:

$$\widehat{F}_{22}^{\infty} = \frac{2-3v}{2(2-v)} \tag{D.1}$$

A similar reasoning for $\widehat{F}_{33}$ yields:

$$\widehat{F}_{33}^{\infty} = \frac{2+v}{2(2-v)} \tag{D.2}$$

Finally, Eqs. (B.2), (B.6) and (17) of Part I yield, since f_{23} is odd:

$$\widehat{F}_{23}^{\infty} \equiv \lim_{p\to+\infty} \widehat{F}_{23}(p) = -\lim_{p\to+\infty} 2\mathrm{i}\int_0^{+\infty} \frac{f_{23}(z)}{z}\cos pz\,\mathrm{d}z + \mathrm{i}\frac{2}{2-v}$$

Now Riemann–Lebesgue's theorem, applied to the function $(f_{23}(z)/z)$ (which is regular at $z = 0$ since f_{23} is odd), implies that the limit here is 0. The value of $\widehat{F}^{\infty}_{23}$ follows:

$$\widehat{F}^{\infty}_{23} = \mathrm{i}\frac{2}{2-\nu} \tag{D.3}$$

Appendix E. Determination of the asymptotic behavior of $f_{22}(z)$, $f_{23}(z)$, $f_{33}(z)$, $g_{22}(z)$, $g_{23}(z)$, $g_{33}(z)$ for $z \to +\infty$

These asymptotic behaviors can be deduced from those of $\widehat{F}_{22}(p)$, $\widehat{F}_{33}(p)$, $\widehat{F}_{23}(p)$, $\hat{g}_{22}(p)$, $\hat{g}_{33}(p)$, $\hat{g}_{23}(p)$ for $p \to 0^+$. Indeed, with regard to $\hat{g}_{22}(p)$ for instance, repeated integration by parts of Eq. (51) yields:

$$\begin{aligned} g_{22}(z) &= \frac{1}{\pi}\left[\hat{g}_{22}(p)\frac{\sin pz}{z}\right]_0^{+\infty} - \frac{1}{\pi}\int_0^{+\infty}\hat{g}'_{22}(p)\frac{\sin pz}{z}\,\mathrm{d}p \\ &= \frac{1}{\pi z}\left[\hat{g}'_{22}(p)\frac{\cos pz}{z}\right]_0^{+\infty} - \frac{1}{\pi z}\int_0^{+\infty}\hat{g}''_{22}(p)\frac{\cos pz}{z}\,\mathrm{d}p \\ &= -\frac{1}{\pi z^2}\left[\hat{g}''_{22}(p)\frac{\sin pz}{z}\right]_0^{+\infty} + \frac{1}{\pi z^2}\int_0^{+\infty}\hat{g}'''_{22}(p)\frac{\sin pz}{z}\,\mathrm{d}p = \frac{1}{\pi z^4}\int_0^{+\infty}\hat{g}'''_{22}(u/z)\sin u\,\mathrm{d}u \quad (pz \equiv u) \end{aligned}$$

The bracketed terms here vanish because of the behavior of $\hat{g}_{22}$ near 0^+ and $+\infty$ (see Sections 4.2 and 4.4). Now Eq. (43) implies that for $p \to 0^+$,

$$\hat{g}'''_{22}(p) \sim \frac{1-2\nu}{2p}$$

Insertion of this result into the preceding expression yields:

$$g_{22}(z) \sim \frac{1-2\nu}{4z^3} \text{ for } z \to +\infty$$

References

Bueckner, H.F., 1987. Weight functions and fundamental fields for the penny-shaped and the half-plane crack in three-space. International Journal of Solids and Structures 23 (1), 57–93.

Gao, H., 1988. Nearly circular shear mode cracks. International Journal of Solids and Structures 24 (2), 177–193.

Gao, H., Rice, J.R., 1986. Shear stress intensity factors for planar crack with slightly curved front. ASME Journal of Applied Mechanics 53 (4), 774–778.

Gao, H., Rice, J.R., 1987a. Nearly circular connections of elastic half spaces. ASME Journal of Applied Mechanics 54 (4), 627–634.

Gao, H., Rice, J.R., 1987b. Somewhat circular tensile cracks. International Journal of Fracture 33 (3), 155–174.

Kassir, M.K., Sih, G., 1975. Three Dimensional Crack Problems. Nordhoff International Publishing, Leyden, The Netherlands.

Lazarus, V., Leblond, J.-B., 1998. Three-dimensional crack-face weight functions for the semi-infinite interface crack. II. Integrodifferential equations on the weight functions and resolution. Journal of the Mechanics and Physics of Solids 46 (3), 513–536.

Leblond, J.-B., Lazarus, V., Mouchrif, S.-E., 1999. Crack paths in three-dimensional elastic solids. II. Three-term expansion of the stress intensity factors—Applications and perspectives. International Journal of Solids and Structures 36 (1), 105–142.

Leblond, J.-B., Mouchrif, S.-E., Perrin, G., 1996. The tensile tunnel-crack with a slightly wavy front. International Journal of Solids and Structures 33 (14), 1995–2022.

Meade, K.P., Keer, L.M., 1984. On the problem of a pair of point forces applied to the faces of a semi-infinite plane crack. Journal of Elasticity 14 (1), 3–14.

Rice, J.R., 1985. First-order variation in elastic fields due to variation in location of a planar crack front. ASME Journal of Applied Mechanics 52 (3), 571–579.

Stallybrass, M., 1981. On the concentrated loading of certain elastic half-space problems and related external crack problems. A new approach. International Journal of Engineering Science 19 (8), 1123–1144.

Tada, H., Paris, P.C., Irwin, G.R., 1973. The Stress Analysis of Cracks Handbook. Del Research Corporation, Hellertown, USA.